ON A CERTAIN MOTION OF A HEAVY SOLID IN THE GORIACHEV-CHAPLYGIN CASE

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A particular case of the solution first obtained by Goriachev [1] and then generalized by Chaplygin [2] is considered. The solution in question is analytically complete [2]. However, it is very difficult to describe the motion of a body in space by means of the formulas expressing the solution of the Goriachev-Chaplygin problem. The authors of [3] and [4] extended the study of body motion, but to the case of rapid rotations of the body only.

We were interested in obtaining a direct kinematic interpretation in this solution on the basis of equations presented in [5] such as that obtained for other particular solutions [6-8] of the problem of motion of a body with a fixed point.

1. The initial expressions. The case of integrability of the equations of motion of a solid about a fixed point considered here was obtained under the conditions $A_1 = A_2 = 4A_3$, $x_2 = x_3 = 0$. By virtue of these conditions the equations of motion can be written as

$$4\frac{d\omega_1}{dt} = 3\omega_2\omega_3, \quad 4\frac{d\omega_2}{dt} = -3\omega_3\omega_1 - a\gamma'', \quad \frac{d\omega_3}{dt} = a\gamma' \tag{1.1}$$

$$\frac{d\gamma}{dt} = \omega_3\gamma' - \omega_2\gamma'', \quad \frac{d\gamma'}{dt} = \omega_1\gamma'' - \omega_3\gamma, \quad \frac{d\gamma''}{dt} = \omega_2\gamma - \omega_1\gamma' \tag{1.2}$$

$$a = Mgx_1 / A_3$$

Let us convert the dimensionless variables in Eqs. (1.1), (1.2). We set

$$\omega_1 = \sqrt{a}\,\omega_1', \quad \omega_2 = \sqrt{a}\,\omega_2', \quad \omega_3 = \sqrt{a}\,\omega_3', \quad t = \frac{t'}{\sqrt{a}} \tag{1.3}$$

Substituting (1.3) into Eqs. (1.1), (1.2) and omitting the primes identifying the dimentionless variables, we obtain

$$4\frac{d\omega_1}{dt} = 3\omega_2\omega_3, \quad 4\frac{d\omega_2}{dt} = -3\omega_3\omega_1 - \gamma'', \quad \frac{d\omega_3}{dt} = \gamma' \tag{1.4}$$

Equations (1.2) remain unchanged.

The integrals of system (1.4), (1.2) are

$$4(\omega_1^2 + \omega_2^2) + \omega_3^2 = 2\gamma + k, \qquad 4(\omega_1\gamma + \omega_2\gamma') + \omega_3\gamma'' = 0 \tag{1.5}$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1, \qquad \omega_3(\omega_1^2 + \omega_2^2) + \omega_1\gamma'' = g_0$$

Let us take ω_1 as the independent variable in terms of which we can express the remaining variables ω_2, ω_3, γ, γ, γ', γ''. We assume that the constant g_0 is equal to zero. Converting to a differentiation with respect to ω_1 in the second equation of system (1.4) with the aid of the first equation of this system and recalling the integral $\omega_3(\omega_1^2 + \omega_2^2) + \omega_1\gamma'' = 0$, we obtain

$$3d \ln(\omega_1^2 + \omega_2^2) = 2d \ln \omega_1$$

Integrating this equation, we find that

$$\omega_2^2 = -\omega_1^2 + b\omega_1^{2/3} \tag{1.6}$$

Here b is an arbitrary positive constant different from zero.

Substituting ω_2^2 from formula (1.6) into relations (1.5), we obtain from the latter the expressions for $\omega_3, \gamma, \gamma', \gamma''$ as functions of ω_1 (requiring that the parameters b and k satisfy the condition $16\, b^3 - k^2 + 4 = 0$)

$$\omega_3^2 = 4\omega_1^{2/3} b^{-1}(\sqrt{4b^3+1}\, \omega_1^{2/3} - 2b^2) \tag{1.7}$$

$$\gamma = b^{-1}(2\sqrt{4b^3+1}\,\omega_1^{4/3} - 2b^2\omega_1^{2/3} - b\sqrt{4b^3+1}) \tag{1.8}$$

$$\omega_2\gamma' = 2\,\omega_1^{1/3} b^{-1}(-\sqrt{4b^3+1}\,\omega_1^2 + b^2\omega_1^{4/3} + b\sqrt{4b^3+1}\,\omega_1^{2/3} - b^3) \tag{1.9}$$

$$\omega_3\gamma'' = -4\omega_1^{1/3}(\sqrt{4b^3+1}\,\omega_1^{2/3} - 2b^2) \tag{1.10}$$

The dependence of ω_1 on t can be determined from the first equation of system (1.4) together with expressions (1.6), (1.7),

$$\frac{d\omega_1}{dt} = \frac{3}{2\sqrt{b}}\,\omega_1^{2/3}[(b - \omega_1^{4/3})(\sqrt{4b^3+1}\,\omega_1^{2/3} - 2b^2)]^{1/2} \tag{1.11}$$

For convenience in investigating the solution of (1.6)–(1.10) we introduce the new variable σ by means of the formula

$$\omega_1 = \sigma\sqrt{\sigma} \tag{1.12}$$

Relations (1.6)–(1.10) and Eq. (1.11) can be written as

$$\omega_2 = \sqrt{\sigma(\sigma^{*2} - \sigma^2)}, \qquad \omega_3 = 2\sigma^*\sqrt{2\sigma_*^{-1}\sigma(\sigma - \sigma_*)} \tag{1.13}$$

$$\gamma = 2\sigma^{*2}(2\sigma^2 - \sigma_*\sigma - \sigma^{*2})/\sigma_*, \qquad \gamma' = 2\sigma^{*2}\sigma_*^{-1}(2\sigma - \sigma_*)\sqrt{\sigma^{*2} - \sigma^2} \tag{1.14}$$

$$\gamma'' = -4\sigma^{*3}\sqrt{1/2\,(\sigma - \sigma_*)\,\sigma_*^{-1}} \tag{1.15}$$

$$d\sigma/dt = \sqrt{2\sigma\sigma^{*-1}(\sigma^{*2} - \sigma^2)(\sigma - \sigma_*)} \tag{1.16}$$

(σ is a elliptic function of time).

Here

$$\sigma_* = \frac{2b^2}{\sqrt{4b^3+1}}, \qquad \sigma^* = \sqrt{b} \tag{1.17}$$

It follows from this that $\sigma_* < \sigma^*$ for all positive values of the parameter b. Solution (1.13)–(1.15) has mechanical meaning as the variable varies in the range

$$\sigma_* \leqslant \sigma \leqslant \sigma^* \tag{1.18}$$

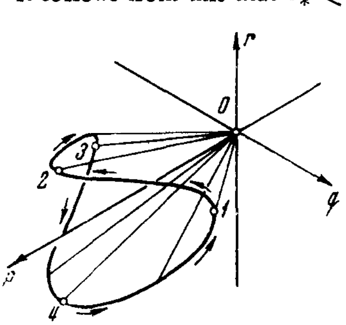

Fig. 1

2. The moving hodograph. The moving hodograph is the trajectory of the extremity of the angular velocity vector in a coordinate system rigidly attached to the body. In our case the moving hodograph is the line of intersection of cylinders (1.6), (1.7). The generatrices of the first cylinder are parallel to the axis $O\omega_3$; those of the second cylinder are parallel to the axis $O\omega_2$. These cylinders intersect in the intervals $\omega_{1*} \leqslant \omega_1 \leqslant \omega_1^*$, $-\omega_1^* \leqslant \omega_1 \leqslant -\omega_{1*}$. We can obtain the values of ω_{1*} and ω_1^* from formula (1.12), by substituting into it the values of σ_* and σ^*, respectively, from expressions (1.17). Figure 1 shows a part of the moving hodograph (the part corresponding to the interval $\omega_{1*} \leqslant \omega_1 \leqslant \omega_1^*$). The other part of the moving hodograph (corresponding to $-\omega_1^* \leqslant \omega_1 \leqslant -\omega_{1*}$) is symmetric to the curve shown with respect to the plane $O\omega_2\omega_3$.

To be specific, let us trace the motion of the extremity of the angular velocity vector (the point M) along the curve shown in Fig. 1. We take σ_* as the initial value of the variable. From (1.12)–(1.15) we infer that at such an instant $\omega_1 = \omega_{1*}$, $\omega_3 = 0$, $\gamma'' = 0$. The latter means that at the initial instant the third coordinate axis is horizontal in fixed space. Let the point M be in position 1 (Fig. 1) at the initial instant $\omega_2 > 0$. Substituting γ' from relations (1.14) into the third equation of system (1.4), we note that $d\omega_3 / dt > 0$ for $\sigma = \sigma_*$. Hence, the point M shifts from position 1 towards position 2, which it reaches in the finite time given by

$$t_* = \int_{\sigma_*}^{\sigma^*} \frac{\sqrt{\sigma^*}\, d\sigma}{\sqrt{2\sigma(\sigma^{*2} - \sigma^2)(\sigma - \sigma_*)}}$$

as determined from Eq. (1.16).

At point 2 we have $\omega_1 = \omega_1^*$, $\omega_2 = 0$, $\omega_3 > 0$, $\gamma' = 0$ (the second coordinate axis of the moving system is horizontal). Substituting ω_1, ω_3, γ'' from relations (1.12), (1.13), (1.15) into the second equation of system (1.4), we find that $d\omega_2 / dt < 0$ for $\sigma = \sigma^*$. Hence, the point M moves from position 2 towards position 3, arriving there in the same time t_*. As above we can show that the point M travels from position 3 to position 4, and then to position 1. The process is then repeated. The point M completes its circuit of the moving hodograph in the time $4t_*$.

3. The fixed hodograph. Interpretation of the body motion. The fixed hodograph is the trajectory of the extremity of the angular velocity vector in fixed space. Following [5], we impose a cylindrical coordinate system of fixed space. The angular velocity vector is defined by the components ω_ζ, ω_ρ, α which can be determined by the relations [5]

$$\omega_\zeta = \omega_1\gamma + \omega_2\gamma' + \omega_3\gamma'' \tag{3.1}$$

$$\omega_\rho^2 = (\omega_2\gamma'' - \omega_3\gamma')^2 + (\omega_3\gamma - \omega_1\gamma'')^2 + (\omega_1\gamma' - \omega_2\gamma)^2 \tag{3.2}$$

$$\omega_\rho^2 \frac{d\alpha}{d\sigma} = \begin{vmatrix} \gamma & \gamma' & \gamma'' \\ \omega_1 & \omega_2 & \omega_3 \\ d\omega_1 / d\sigma & d\omega_2 / d\sigma & d\omega_3 / d\sigma \end{vmatrix} \tag{3.3}$$

Let us substitute expressions (1.12)–(1.15) into Eqs. (3.1)–(3.3),

$$\omega_\zeta = -6\sigma_*^{-1}\sigma^{*4} \sqrt{\sigma}\,(\sigma - \sigma_*) \tag{3.4}$$

$$\omega_\rho^2 = 36\, \sigma_*^{-2} \sigma^{*8} \sigma (\sigma - \sigma_1)(\sigma_2 - \sigma) \tag{3.5}$$

$$\frac{d\alpha}{d\sigma} = \frac{\sqrt{\sigma_*}\,(\sigma - \sigma_3)(\sigma + \sigma_4)}{3\sigma^{*3}(\sigma_1 - \sigma)(\sigma - \sigma_2)\sqrt{2(\sigma^{*2} - \sigma^2)(\sigma - \sigma_*)}} \tag{3.6}$$

where

$$\sigma_{1,2} = {}^1/_{18}\, \sigma_*\sigma^{*-6}\,[2(1 + 9\sigma^*)^6 \pm \sqrt{4 + 9\sigma^{*6}}] \tag{3.7}$$

$$\sigma_3 = {}^1/_6\,(3\sigma_* + \sqrt{3(4\sigma^{*2} - \sigma_*^2)}), \qquad \sigma_4 = {}^1/_6\,(-3\sigma_* + \sqrt{3(4\sigma^{*2} - \sigma_*^2)}) \tag{3.8}$$

Let us investigate the meridian of the surface of revolution defined by Eqs. (3.4), (3.5) in the plane $O\omega_\rho\omega_\zeta$ as σ varies in interval (1.18).

From relations (3.4), (3.5) we have

$$\frac{d\omega_\zeta}{d\omega_\rho} = -\frac{(3\sigma - \sigma_*)\sqrt{(\sigma_1 - \sigma)(\sigma - \sigma_2)}}{3(\sigma - \sigma_6)(\sigma_5 - \sigma)} \tag{3.9}$$

Here

$$\sigma_{5,6} = {}^1/_{54}\, \sigma_*\sigma^{*-6}\,[4(1 + 9\sigma^{*6}) \pm \sqrt{324\sigma^{*12} + 99\sigma^{*6} + 16}\,] \tag{3.10}$$

Comparing the quantities σ_1 and σ_2 from (3.7) with σ_* and σ^*, we find that the inequalities $0 < \sigma_2 < \sigma_*$, $\sigma_1 > \sigma^*$ hold for all values of the parameter $b > 0$. Hence, ω_ρ^2 does not vanish in interval (1.18). From formula (3.9) we infer that the tangent to the meridian is not parallel to the axis $O\omega_\rho$ anywhere in the interval $[\sigma_*, \sigma^*]$.

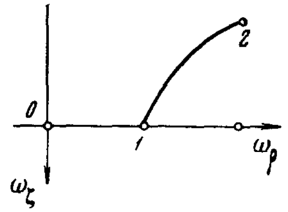

Fig. 2

Since $\sigma = 0$ is the root of the equation $\omega_\rho^2 = 0$ and since $0 < \sigma_2 < \sigma_*$, it follows that the root σ_6 of the equation $d\omega_\rho / d\sigma = 0$ lies between zero and σ_2.

The value of σ_5 from (3.10) also lies outside interval (1.18) for all $b > 0$. In order to show this we note, first of all, that $\sigma_5 > \sigma_*$. Next, substituting the values of σ_* and σ^* from (1.17) into the denominator of formula (3.9), we find that $d\omega_\rho / d\sigma > 0$ for $\sigma = \sigma_*, \sigma^*$. Hence, ω_ρ increases in interval (1.18) and $\sigma_5 > \sigma^*$. Moreover, Eq. (3.9) implies that the derivatives $d\omega_\zeta / d\sigma_\rho$ and $d^2\omega_\zeta / d\omega_\rho^2$ are negative as σ varies in (1.18).

The meridian is of the form shown in Fig. 2. Rotating this curve about the axis $O\omega_\zeta$, we obtain that part of the surface of revolution on which the fixed hodograph lies.

Let us investigate the projection of the fixed hodograph on the horizontal plane of the fixed coordinate system $O\xi\eta$. To this end we trace the variation of the functions ω_ρ and α as functions of σ in this plane. We have already shown that as σ varies from σ_* to σ^*, the function ω_ρ increases from $\omega_\rho(\sigma_*)$ to $\omega_\rho(\sigma^*)$. The projection of the fixed hodograph lies between the two circles of radii $\omega_\rho(\sigma_*)$, $\omega_\rho(\sigma^*)$.

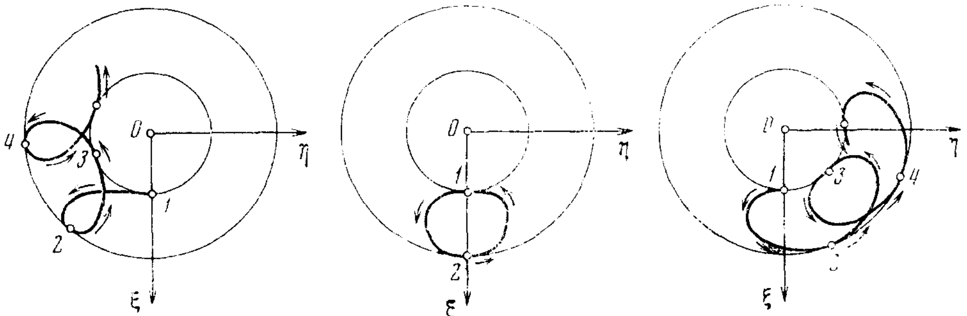

Fig. 3

Let us denote by θ the angle between the direction of the radius ω_ρ and the tangent to the projection at the given point, measures counterclockwise,

$$\operatorname{tg} \theta = \frac{\omega_\rho d\alpha}{d\omega_\rho}$$

Substituting expressions (3.5), (3.6) into this equation we find that in interval (1.18) the projection of the fixed hodograph on the plane does not meet the circles $\omega_\rho = \text{const}$ at a right angle; moreover, it approaches and osculates the boundary circles $\omega_\rho(\sigma_*)$, $\omega_\rho(\sigma^*)$.

We choose the axis $O\xi$ in such a way that $\alpha = 0$ for $\sigma = \sigma_*$. Since σ_* is the minimum value of the variable, then σ increases with time, so that the radical in Eq. (1.16) is positive in this case. Recalling this fact, we infer from formula (3.6) that

$$\alpha = \int_{\sigma_*}^{\sigma} \frac{\sqrt{\sigma_*}(\sigma - \sigma_3)(\sigma + \sigma_4)\, d\sigma}{3\sigma^{*3}(\sigma_1 - \sigma)(\sigma - \sigma_2)\sqrt{2(\sigma^{*2} - \sigma^2)(\sigma - \sigma_*)}} \tag{3.11}$$

From Eqs. (1.16), (3.6) we infer that the sign of the derivative $d\alpha / dt$ depends on the sign of the expression $\sigma - \sigma_3$. We note that for all $b > 0$ the root σ_3 lies between the boundary values σ_* and σ^*. Hence, the difference $\sigma - \sigma_3$ is negative at the initial instant. The angle α begins to decrease. This continues until the variable σ reaches the value σ_3. Once $d\alpha / dt > 0$ $(\sigma - \sigma_3 > 0)$, the angle begins to increase to the value α_0, which we obtain from (3.11) when the upper limit is equal to σ^*.

Computing integral (3.11) numerically, we conclude that α_0 differs for different values of b: when b satisfies the inequality $0 < b < b^* \approx 0.653182$ we have $\alpha_0 < 0$; for $b = b^*$ we have $\alpha_0 = 0$; for $b > b^*$ we have $\alpha_0 > 0$. Thus, as σ varies from σ_* to σ^* the projection of the moving hodograph constitutes part of the curve shown in Fig. 3: 3a for $b < b^*$, 3b for $b = b^*$, 3c for $b > b^*$ lying between points *1* and *2*.

On reaching the value σ^* the variable σ must begin to decrease, which means that $d\sigma / dt < 0$, and consequently the radical $\sqrt{2(\sigma^{*2} - \sigma^2)(\sigma - \sigma_*)}$ in formulas (1.6), (3.11) changed sign. As σ varies (from σ^* to σ_*) the functions ω_p and $d\alpha / dt$ assume their previous values in the opposite direction. The portion of the projection which corresponds to this time interval is symmetric to the curve investigated above with respect to the ray α_0. In Fig. 3 it lies between points *2* and *3*. In the same way we can construct the projection of the fixed hodograph on the plane $O\xi\eta$ with further variation of σ (Fig. 3).

The cylinder with vertical generatrices whose directrix is the curve shown in Fig. 3 intersects the surface of revolution to yield the fixed hodograph of angular velocity (Fig. 4).

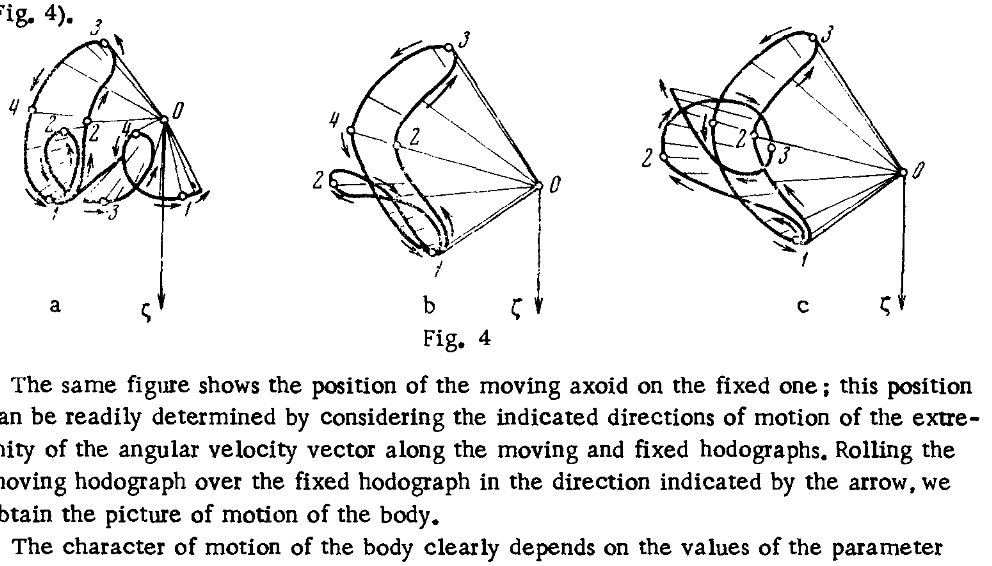

Fig. 4

The same figure shows the position of the moving axoid on the fixed one; this position can be readily determined by considering the indicated directions of motion of the extremity of the angular velocity vector along the moving and fixed hodographs. Rolling the moving hodograph over the fixed hodograph in the direction indicated by the arrow, we obtain the picture of motion of the body.

The character of motion of the body clearly depends on the values of the parameter b: if $b < b^*$, the body precesses clockwise about the vertical (Fig. 4a); for $b = b^*$ it executes a periodic motion and the moving and fixed hodographs are closed curves (Fig. 4b); if $b > b^*$, the body precesses counterclockwise about the vertical (Fig. 4c).

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ON A CERTAIN SOLUTION OF THE PROBLEM OF MOTION OF A GYROSCOPE ON GIMBALS

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The equations of motion of a heavy gyroscope on gimbals are integrated for an arbitrary position of the center of mass of the gyro housing.

In 1958 Chetaev [1] investigated the motion of a heavy gyro on gimbals in the case of vertical position of the outer gimbal axis of rotation (output axis). The center of gravity of the housing and gyro was assumed to coincide with the axis of symmetry of the gyro. Chetaev reduced the problem of integrating the equations of motion to quadratures. These quadratures can be readily extended to the case where the gyroscope is acted along the axis of rotation of its housing by a moment of external forces which is an arbitrary integrable function of the angle of nutation.

This problem was considered in [2] under certain assumptions concerning the moments of inertia of the system elements and for certain specific initial data.

1. Let us consider a gyro on gimbals under the assumption that the fixed axis of rotation of the outer gimbal is in vertical position. We introduce two right-handed coordinate systems with a common origin at a fixed point O of the gyroscope. The axis ξ_3 of the fixed coordinate system ξ_1, ξ_2, ξ_3 is directed vertically upward along the axis of rotation of the outer gimbal; the axes ξ_1 and ξ_2 lie in the horizontal plane. The axes η_1 and η_2 of the moving coordinate system $\eta_1\eta_2\eta_3$ (which is rigidly attached to the gyroscope housing) are directed along the axis of rotation of the housing and along the axis of symmetry of the gyro, respectively. The position of the system under consideration in the space $\xi_1\xi_2\xi_3$ is defined by the three Euler angles, namely the angle of precession ψ, the angle of nutation θ, and the angle of proper rotation φ of the gyro relative to the